Solving the radiation equation

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Heat can transfer at the speed of light through radiation. **The Stefan-Boltzmann Law** describes how the power E emitted from an object depends on its temperature:

$$E = \varepsilon\sigma T^4 \qquad (1)$$

where σ is called the Stefan constant, ε is the emissivity (1 if the object is a perfect black body), and T is the temperature in Kelvin.

For a system that has many objects, we have to discretize every one of them into many small patches $\{dA_i\}$ in order to model their interactions. The power density given away by each patch is called **radiosity** (denoted by B), which is governed by the following integral equation (this is in fact an inhomogeneous Fredholm equation of the second kind[1]):

$$B_i dA_i = E_i dA_i + \rho_i \int_{\Omega} B_j F_{ji} V_{ji} dA_j \qquad (2)$$

where B_i is the outgoing power per unit area from patch i, E_i is the emission power per unit area of the patch, ρ_i is the reflectivity of the patch, dA_i is the area of the patch, F_{ji} is the view factor (dimensionless), and V_{ji} is the visibility function (0 if patches i and j cannot see each other, otherwise 1). The visibility function can be calculated using collision detection between rays and patches.

The view factor from a surface A_1 to a surface A_2 defines the proportion of the radiation leaving A_1 that ends up striking A_2. In 3D, it can be calculated through the following formula based on the etendue of a ray:

$$F_{12} = \frac{1}{A_1} \int_{A_1} \int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\pi r_{12}^2} dA_1\, dA_2 \qquad (3)$$

where r_{12} is the distance between the two surfaces, and θ_1 and θ_2 are the acute angles between the surface normal and the distance vector. In 2D, it can be calculated by replacing the areas with lengths and r_{12}^2 with r_{12}:

$$F_{12} = \frac{1}{L_1} \int_{L_1} \int_{L_2} \frac{\cos\theta_1 \cos\theta_2}{\pi r_{12}} dL_1\, dL_2 \qquad (3a)$$

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[1] http://en.wikipedia.org/wiki/Fredholm_integral_equation

The net energy that converts into thermal energy at a patch comes from the difference between the portion of energy reflected from all other patches that becomes absorbed by the patch and the energy that it emits:

$$Q_i dA_i = \alpha_i \int_{\Omega} B_j F_{ji} V_{ji} dA_j - E_i dA_i \qquad (4)$$

where α_i is the absorptivity (the ability to absorb electromagnetic radiation) of the patch's material. This equation can be coupled with the Heat Equation and the Navier-Stokes Equation to allow radiation to interact with the flow of heat and mass.

Equations (2) and (4) can be discretized into the following matrix equations:

$$B_i - \rho_i \sum_j F_{ij} V_{ij} B_j = E_i \qquad (5)$$

$$Q_i = \alpha_i \sum_j F_{ij} V_{ij} B_j - E_i \qquad (6)$$

Equation (5) can be solved using a relaxation method, such as the Gauss-Seidel method,[2] using the emission power at each patch as the initial guess. In a transient simulation, only a few relaxation steps are needed as the time-dependent solution is already somehow iterative.

The thermal power generated at each patch Q_i can then be derived using Equation (6).

[2] http://en.wikipedia.org/wiki/Gauss%E2%80%93Seidel_method