Solving the radiation equation

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Heat can transfer at the speed of light through radiation. The Stefan-Boltzmann Law describes how the power $E$ emitted from an object depends on its temperature:

$$E = \varepsilon\sigma T^4$$

where $\sigma$ is called the Stefan constant, $\varepsilon$ is the emissivity (1 if the object is a perfect black body), and $T$ is the temperature in Kelvin.

For a system that has many objects, we have to discretize every one of them into many small patches $\{dA_i\}$ in order to model their interactions. The power density given away by each patch is called radiosity (denoted by $B$), which is governed by the following integral equation (this is in fact an inhomogeneous Fredholm equation of the second kind$^1$):

$$B_i dA_i = E_i dA_i + \rho_i \int \hat{n}_i B_j F_{ji} V_{ji} dA_j$$

where $B_i$ is the outgoing power per unit area from patch $i$, $E_i$ is the emission power per unit area of the patch, $\rho_i$ is the reflectivity of the patch, $dA_i$ is the area of the patch, $F_{ji}$ is the view factor (dimensionless), and $V_{ji}$ is the visibility function (0 if patches $i$ and $j$ cannot see each other, otherwise 1). The visibility function can be calculated using collision detection between rays and patches.

The view factor from a surface $A_1$ to a surface $A_2$ defines the proportion of the radiation leaving $A_1$ that ends up striking $A_2$. In 3D, it can be calculated through the following formula based on the etendue of a ray:

$$F_{12} = \frac{1}{A_1} \int_{A_2} \frac{\cos \theta_1 \cos \theta_2}{\pi r_{12}^2} dA_2 dA_1$$

where $r_{12}$ is the distance between the two surfaces, and $\theta_1$ and $\theta_2$ are the acute angles between the surface normal and the distance vector. In 2D, it can be calculated by replacing the areas with lengths and $r_{12}^2$ with $r_{12}$:

$$F_{12} = \frac{1}{L_1} \int_{L_2} \frac{\cos \theta_1 \cos \theta_2}{\pi r_{12}} dL_1 dL_2$$

$^1$ http://en.wikipedia.org/wiki/Fredholm_integral_equation
The net energy that converts into thermal energy at a patch comes from the difference between the portion of energy reflected from all other patches that becomes absorbed by the patch and the energy that it emits:

\[ Q_i dA_i = \alpha_i \int \sum_{j} B_j F_{ij} V_{ij} dA_j - E_i dA_i \quad (4) \]

where \( \alpha_i \) is the absorptivity (the ability to absorb electromagnetic radiation) of the patch’s material. This equation can be coupled with the Heat Equation and the Navier-Stokes Equation to allow radiation to interact with the flow of heat and mass.

Equations (2) and (4) can be discretized into the following matrix equations:

\[ B_i - \rho_i \sum_j F_{ij} V_{ij} B_j = E_i \quad (5) \]

\[ Q_i = \alpha_i \sum_j F_{ij} V_{ij} B_j - E_i \quad (6) \]

Equation (5) can be solved using a relaxation method, such as the Gauss-Seidel method,\(^2\) using the emission power at each patch as the initial guess. In a transient simulation, only a few relaxation steps are needed as the time-dependent solution is already somehow iterative.

The thermal power generated at each patch \( Q_i \) can then be derived using Equation (6).

\(^2\) http://en.wikipedia.org/wiki/Gauss%E2%80%93Seidel_method